Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Geometry and Physics 48 (2003) 12–43
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Painlevé expansions and the Einstein equations: the two-summand case

Andrew Dancer [a,*], McKenzie Y. Wang [b]

[a] *Jesus College, Oxford University, Oxford OX1 3DW, UK*
[b] *Department of Mathematics and Statistics, McMaster University, Hamilton, Ont., Canada L8S 4K1*

Received 22 November 2002

Abstract

We investigate the existence of Painlevé–Kovalevskaya expansions for various reductions to ordinary differential equations of the Ricci-flat equations. We investigate links between such expansions and metrics of exceptional holonomy.
© 2003 Elsevier Science B.V. All rights reserved.

MSC: 83C05

Subj. Class.: General relativity

Keywords: Painlevé expansions; Einstein equations

1. Introduction

One tool in the study of differential equations is to investigate the existence of families of meromorphic solutions. This technique goes back to Kovalevskaya's work on integrable tops [9] and to Painlevé's work on movable singularities of solutions to differential equations. For more recent work on this subject, see, for example [1,2].

In this paper we shall investigate the existence of such meromorphic expansions for the cohomogeneity one Ricci-flat Einstein equations when the isotropy representation of the principal orbit consists of two inequivalent summands. Two rather special cases were analysed in [6], including the situation of double warped product metrics. For these metrics, we found that larger families of Painlevé expansions existed in the 10 and 11-dimensional cases than in other dimensions. These were exactly the dimensions where, in some cases, conserved quantities for the equations were found in [4].

* Corresponding author.
E-mail addresses: dancer@maths.ox.ac.uk (A. Dancer), wang@mcmail.mcmaster.ca (M.Y. Wang).

0393-0440/$ – see front matter © 2003 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0393-0440(03)00023-8

Some of the themes of the analysis in [6] recur in the current paper. We find that the existence and size of families of such Painlevé expansions depend rather sensitively on the choice of principal orbit. In particular, certain features (such as the existence of certain types of Painlevé expansion) occur only if the dimension of one or both of the summands in the isotropy representation is small.

As in [6], existence of a non-trivial family is often associated to the existence of a solution to a Diophantine equation (such as the presence of an integral point on an elliptic curve), and this may single out certain dimensions as special.

Finally, we find that Painlevé expansions are sometimes linked to the existence of a subsystem of the Ricci-flat equations representing metrics of exceptional holonomy, cf. Examples 7.1 and 7.4.

The layout of the paper is as follows. In Sections 2 and 3 we choose variables so as to put the Einstein system into a form suitable for Painlevé analysis. In Section 4 we begin our study of the case when the principal orbit G/K has two distinct summands in its isotropy representation and K is not maximal in G. We find the possible leading terms of a Painlevé expansion (as in [6] we allow expansions which are meromorphic in a fractional power of the independent variable). Next, in Section 5 we substitute the expansion into the equations and find the recursion relations that the coefficients must satisfy. We compute the resonances, that is, the steps in the recursion at which free parameters may enter. These are the steps at which the linear operator in the recursion fails to be invertible. Existence of Painlevé expansions depending on a large number of parameters requires there to be many rational resonances, and this often leads to Diophantine constraints on the parameters in the equations. In Section 6 we study the compatibility conditions for the recursion at the resonances to be solvable. Section 7 is devoted to examples. In Section 8 we perform a similar analysis for the case when K is maximal in G. Lastly, we describe in Section 9 the asymptotic behaviour of the Ricci-flat metrics corresponding to our Painlevé expansions.

2. The Einstein equations and an associated quadratic system

We consider the Einstein equations for a cohomogeneity one metric $\bar{g}$ with principal orbit G/K, where the isotropy representation is *monotypic*; that is we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r,$$

where the summands $\mathfrak{p}_i$ are *inequivalent* K-modules of real dimension d_i. The metric $\bar{g}$ may be written as $\mathrm{d}t^2 + g_t$, where

$$g_t = \mathrm{e}^{q_1(t)} B|_{\mathfrak{p}_1} \perp \cdots \perp \mathrm{e}^{q_r(t)} B|_{\mathfrak{p}_r}, \tag{2.1}$$

and B is a background metric on G/K induced by some bi-invariant metric on G. We use d to denote the vector of dimensions $(d_1, \ldots, d_r)$, and let $n = \sum_{i=1}^{r} d_i$ denote the dimension of the principal orbit. The cohomogeneity one Einstein metric therefore lives on a space of dimension $n + 1$.

As explained in [5] the Einstein equations $\mathrm{Ric}(\bar{g}) = \Lambda \bar{g}$ may be written as a Hamiltonian flow together with the constraint $H = 0$ on the cotangent space of the space of

G-invariant metrics on G/K. Writing $q = (q_1, \dots, q_r)$ and $p = (p_1, \dots, p_r)$, where p_i are the associated momentum variables, the Hamiltonian is

$$H = \mathrm{e}^{-(1/2)d\cdot q} pJp^{\mathrm{T}} + \mathrm{e}^{(1/2)d\cdot q} \left((n-1)\Lambda - \sum_{j=1}^{r+m} A_j \,\mathrm{e}^{w^{(j)}\cdot q} \right). \tag{2.2}$$

In the above, J is the symmetric matrix with entries

$$J_{ii} = \frac{1}{n-1} - \frac{1}{d_i}, \qquad J_{ij} = \frac{1}{n-1} \quad \text{for } i \neq j,$$

and so it has one positive and $r-1$ negative eigenvalues. Furthermore, the A_j are constants, the $w^{(j)}$ are vectors in $\mathbb{Z}^r$, and the term $\sum_{j=1}^{r+m} A_j \,\mathrm{e}^{w^{(j)}\cdot q}$ is the scalar curvature of the metric (2.1) on G/K. It follows from the scalar curvature formula of a homogeneous metric in [10] that the vectors $w^{(j)}$ may be of three kinds:

(i) one entry is -1, the rest are zero;
(ii) one entry is 1, two are -1, the rest are zero;
(iii) one entry is 1, one is -2, the rest are zero.

In particular, $w^{(j)} \cdot (1, \dots, 1) = -1$ in all cases. We denote by $w_i^{(j)}$ the ith entry of $w^{(j)}$.

In order to carry out the Painlevé analysis, it will be advantageous to replace Hamilton's equations for the Hamiltonian H by a quadratic system involving $2r + m$ new dependent variables and m additional constraints. As well as simplifying calculations, this has the advantage that general arguments about systems with only quadratic non-linearities guarantee that the formal series solutions we construct are in fact convergent on a punctured neighbourhood of the singularity (cf. [2,6]).

A special case ($m = 0$) of this transformation was already used in [6], where we adapted to our situation a similar transformation of the Toda-lattice equations discussed by Adler and van Moerbeke [2].

We now explain how the transformation works in the general situation, i.e., when $m \geq 0$. Let C be a matrix such that $C^{-1}J(C^{-1})^{\mathrm{T}} = \mathrm{diag}(\mu_1, \dots, \mu_r)$ and introduce new symplectic coordinates a, b by $q = Ca$, $b = pC$, and set $\bar{d} = dC$, $\bar{w}^{(j)} = w^{(j)}C$. Taking a new coordinate s defined by $\mathrm{d}s = \mathrm{e}^{-(1/2)\bar{d}\cdot a}\,\mathrm{d}t$, we saw in [6] that the Einstein equations are equivalent to the Hamiltonian flow for the new Hamiltonian

$$\bar{H} = \mathrm{e}^{(1/2)\bar{d}\cdot a} H = \sum_{j=1}^{r} \mu_j b_j^2 + (n-1)\Lambda\,\mathrm{e}^{\bar{d}\cdot a} - \sum_{j=1}^{r+m} A_j \,\mathrm{e}^{(\bar{d}+\bar{w}^{(j)})\cdot a}. \tag{2.3}$$

We shall assume that the set of vectors $w^{(j)}$ contains a basis for $\mathbb{R}^r$ (this is always true if G is semisimple, cf. the proof of Theorem 3.11 in [5]), and by reordering we take $w^{(1)}, \dots, w^{(r)}$ to be such a basis. We define an $m \times r$ matrix ν by

$$w^{(i+r)} = \sum_{j=1}^{r} \nu_{ij} w^{(j)} \quad (1 \leq i \leq m).$$

Notice that taking the scalar product of both sides with $(1, \dots, 1)$ shows that

$$\sum_{j=1}^{r} \nu_{ij} = 1 \quad \text{for all } i.$$

We now introduce a new set of symplectic coordinates by setting

$$x_i = \mathrm{e}^{(\bar{d}+\bar{w}^{(i)})\cdot a} = \mathrm{e}^{(d+w^{(i)})\cdot q} \quad (i = 1, \dots, r),$$

and y_i to be the associated momentum variables. Let U be the invertible $r \times r$ matrix defined by $U_{ij} = d_j + w_j^{(i)}$ $(1 \le i, j \le r)$ and let $E = UJU^{\mathrm{T}}$. If we also define $\bar{U} = UC$, then the momentum variables satisfy

$$y_i = \sum_j b_j \bar{U}^{ji} x_i^{-1},$$

so $b = (x_1 y_1, \dots, x_r y_r)\bar{U}$.

Moreover,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \bar{U}^{-1} \begin{pmatrix} \log x_1 \\ \vdots \\ \log x_r \end{pmatrix},$$

so setting $\xi = -\bar{d}\bar{U}^{-1}$, we have

$$\sum_{i=1}^{r} \xi_i \log x_i = -\bar{d}a.$$

The new Hamiltonian now becomes

$$\begin{aligned} \bar{H} &= \sum_{i,j=1}^{r} E_{ij} x_i y_i x_j y_j + (n-1)\Lambda\, \mathrm{e}^{\bar{d}\cdot a} - \sum_{j=1}^{r+m} A_j\, \mathrm{e}^{(\bar{d}+\bar{w}^{(j)})\cdot a} \\ &= \sum_{i,j=1}^{r} E_{ij} x_i y_i x_j y_j + (n-1)\Lambda \prod_{j=1}^{r} x_j^{-\xi_j} - \sum_{j=1}^{r} A_j x_j - \sum_{j=1}^{m} A_{j+r} \prod_{k=1}^{r} x_k^{\nu_{jk}}. \end{aligned}$$

Hamilton's equations for $\bar{H}$ in these variables are

$$x_i' = 2x_i \sum_{j=1}^{r} E_{ij} x_j y_j,$$

$$y_i' = -\left(2y_i \sum_{j=1}^{r} E_{ij} x_j y_j - (n-1)\Lambda \xi_i x_i^{-1} \prod_{j=1}^{r} x_j^{-\xi_j} - A_i - \sum_{j=1}^{m} A_{j+r} \nu_{ji} x_i^{-1} \prod_{k=1}^{r} x_k^{\nu_{jk}} \right)$$

for $i = 1, \dots, r$. Letting $u_i = x_i y_i$ we can rewrite this as

$$x_i' = 2x_i \sum_{j=1}^{r} E_{ij} u_j = 2x_i (Eu)_i, \tag{2.4}$$

$$u_i' = A_i x_i + \sum_{j=1}^{m} A_{j+r} \nu_{ji} \prod_{k=1}^{r} x_k^{\nu_{jk}} + (n-1)\Lambda \xi_i \prod_{j=1}^{r} x_j^{-\xi_j}. \tag{2.5}$$

We now specialise to the case $\Lambda = 0$. We set

$$x_{j+r} = \mathrm{e}^{(\bar{d} + \bar{w}^{(j+r)}) \cdot a} = \prod_{k=1}^{r} x_k^{\nu_{jk}} \quad (1 \le j \le m), \tag{2.6}$$

so that

$$x_{j+r}' = x_{j+r} \sum_{k=1}^{r} \nu_{jk} x_k^{-1} x_k' = 2x_{j+r} (\nu E u)_j. \tag{2.7}$$

Introduce a matrix $\hat{U}$ defined by

$$\hat{U}_{ij} = d_j + w_j^{(i)} \quad (1 \le i \le r+m, 1 \le j \le r),$$

so that

$$\hat{U} = \begin{pmatrix} U \\ \nu U \end{pmatrix},$$

and hence

$$(-\nu \quad I_m) \hat{U} = 0. \tag{2.8}$$

Also $\hat{U} J \hat{U}^{\mathrm{T}}$ is the $(r+m) \times (r+m)$ matrix whose ijth entry is $J(d + w^{(i)}, d + w^{(j)})$. We can write

$$\hat{U} J \hat{U}^{\mathrm{T}} = \Theta \hat{D} \Theta^{\mathrm{T}},$$

where

$$\Theta = \begin{pmatrix} \Theta_1 & 0_{r \times m} \\ \Theta_2 & 0_{m \times m} \end{pmatrix}, \qquad \hat{D} = \begin{pmatrix} D & 0_{r \times m} \\ 0_{m \times r} & 0_{m \times m} \end{pmatrix},$$

D is the $r \times r$ matrix $\mathrm{diag}(1, -1, \dots, -1)$, Θ_1 is $r \times r$ and of rank r, Θ_2 is $m \times r$, and $E = \Theta_1 D \Theta_1^{\mathrm{T}}$. Observe that

$$\Theta \hat{D} \Theta^{\mathrm{T}} \begin{pmatrix} -\nu^{\mathrm{T}} \\ I_m \end{pmatrix} = \hat{U} J \left(\hat{U}^{\mathrm{T}} \begin{pmatrix} -\nu^{\mathrm{T}} \\ I_m \end{pmatrix} \right) = 0$$

from (2.8). But also

$$\hat{D}\Theta^{\mathrm{T}}\begin{pmatrix}-\nu^{\mathrm{T}}\\ I_m\end{pmatrix}=\begin{pmatrix}D\Theta_1^{\mathrm{T}} & D\Theta_2^{\mathrm{T}}\\ 0 & 0\end{pmatrix}\begin{pmatrix}-\nu^{\mathrm{T}}\\ I_m\end{pmatrix}=\begin{pmatrix}-D\Theta_1^{\mathrm{T}}\nu^{\mathrm{T}}+D\Theta_2^{\mathrm{T}}\\ 0\end{pmatrix},$$

which can only be in the kernel of Θ if it is zero, that is, if $\Theta_2^{\mathrm{T}}=\Theta_1^{\mathrm{T}}\nu^{\mathrm{T}}$, and hence

$$\hat{D}\Theta^{\mathrm{T}}\begin{pmatrix}\nu^{\mathrm{T}}\\ 0\end{pmatrix}=\hat{D}\Theta^{\mathrm{T}}\begin{pmatrix}0\\ I_m\end{pmatrix}. \tag{2.9}$$

Note that we have shown that $\Theta_2=\nu\Theta_1$.

Let $z_i=A_ix_i$ and define

$$v=\hat{D}\Theta^{\mathrm{T}}\begin{pmatrix}u\\ 0_{m\times 1}\end{pmatrix},$$

where u is the column vector consisting of the u_i. It follows that $v_{r+1}=\cdots=v_{r+m}=0$. Then from (2.4) and (2.7) we have $z_i'=2z_i(\Theta v)_i$ and from (2.5) we obtain

$$v'=\hat{D}\Theta^{\mathrm{T}}\begin{pmatrix}u'\\ 0_{m\times 1}\end{pmatrix}=\hat{D}\Theta^{\mathrm{T}}\left(\begin{pmatrix}z_1\\ \vdots\\ z_r\\ 0_{m\times 1}\end{pmatrix}+\begin{pmatrix}\nu^{\mathrm{T}}\\ 0_{m\times m}\end{pmatrix}\begin{pmatrix}z_{r+1}\\ \vdots\\ z_{r+m}\end{pmatrix}\right).$$

Using (2.9), we finally obtain the following quadratic system:

$$z_i'=2z_i\sum_{j=1}^{r+m}\Theta_{ij}v_j, \tag{2.10}$$

$$v_i'=\epsilon_i\sum_{j=1}^{r+m}\Theta_{ji}z_j, \tag{2.11}$$

where $\epsilon_1=1$, $\epsilon_i=-1$ for $1<i\le r$, and $\epsilon_i=0$ for $i>r$.

Observe that the Hamiltonian $\bar{H}$ can now be written as

$$\bar{H}=v_1^2-v_2^2-\cdots-v_r^2-\sum_{j=1}^{r+m}z_j,$$

and the relations (2.6) may be viewed as additional constraints

$$\frac{z_{r+j}}{A_{r+j}}=\prod_{i=1}^{r}\left(\frac{z_i}{A_i}\right)^{\nu_{ji}}\quad (1\le j\le m), \tag{2.12}$$

which, together with the Hamiltonian constraint $\bar{H}=0$, single out the solutions which solve the original Einstein system.

Remark 2.1. On the other hand, if we are given the system (2.10) and (2.11), then using the property (2.9), it follows easily that:

$$\left(\prod_{j=1}^{r} |z_j|^{\nu_{ij}}\right) |z_{i+r}|^{-1}$$

are first integrals of the system.

In fact, we can define a Poisson structure on

$$\mathbb{R}_{\pm}^{r+m} \times \mathbb{R}^{r+m} = \{((z_1, \dots, z_{r+m}), (v_1, \dots, v_{r+m})), z_i \in \mathbb{R} - \{0\}, v_i \in \mathbb{R}, \}$$

by introducing the bivector

$$\Omega = \sum_{i,j=1}^{r+m} \Omega^{ij} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial z_j},$$

where $\Omega^{ij} = \epsilon_i \Theta_{ji} z_j$, and using it to define the Poisson bracket $\{F_1, F_2\}$ to be equal to

$$\sum_{i,j} \Omega^{ij} \left(\frac{\partial F_1}{\partial v_i} \frac{\partial F_2}{\partial z_j} - \frac{\partial F_1}{\partial z_j} \frac{\partial F_2}{\partial v_i} \right).$$

Then the Hamiltonian vector field corresponding to the function $\bar{H}$ under this Poisson structure is equivalent to the system (2.10) and (2.11). Furthermore, the variety defined by

$$\mathcal{L} = \left\{ v_{r+j} = 0, \frac{z_{r+j}}{A_{r+j}} = \prod_{i=1}^{r} \left(\frac{z_i}{A_i}\right)^{\nu_{ji}} \quad (1 \leq j \leq m) \right\}$$

is a symplectic leaf of the above Poisson manifold and the Hamiltonian flow of $\bar{H}$ on $\mathcal{L} \cap \{\bar{H} = 0\}$ is equivalent to the cohomogeneity one Einstein system. It follows immediately that an integral curve of the system (2.10) and (2.11) which starts in $\mathcal{L} \cap \{\bar{H} = 0\}$ remains in it for all time.

Remark 2.2. If $\Lambda \neq 0$, since $d = -\xi U$, we may add $-\xi$ to the last row of ν, regard $(1-n)\Lambda$ to be another constant A_0, and hence incorporate the term $(n-1)\Lambda\, \mathrm{e}^{\bar{d}\cdot a}$ into the scalar curvature formula.

The above discussion also applies to the case when we have a Lorentz metric and the principal orbits are space-like hypersurfaces. We simply need to replace all the constants A_i (including A_0 if $\Lambda \neq 0$) by $-A_i$ and $\mathrm{d}t^2$ in $\bar{g}$ by $-\mathrm{d}t^2$.

3. The two-summand case

Let us specialise to the case where $r = 2$, that is, there are two inequivalent summands in the isotropy representation. Recall that the principal orbit G/K is an almost effective connected compact homogeneous space, where G is a compact Lie group and K is a closed subgroup, neither of which is assumed to be connected. According to the calculations leading to (1.3) in [10], the scalar curvature of a G-invariant metric on G/K takes the form

$$\mathsf{S} = -A_1\,e^{q_1} - A_2\,e^{q_2} + A_3\,e^{q_1-2q_2} + A_4\,e^{q_2-2q_1}, \tag{3.1}$$

where A_1 and A_2 are non-negative, $A_3 = -(1/4)\Gamma_{221}$, $A_4 = -(1/4)\Gamma_{112}$, and the constant Γ_{ijk} denotes the sum

$$\sum_{\alpha,\beta,\gamma} B([e_\alpha, e_\beta], e_\gamma)^2$$

in which $\{e_\alpha\}$, $\{e_\beta\}$, and $\{e_\gamma\}$, respectively, range over B-orthonormal bases of $\mathfrak{p}_i, \mathfrak{p}_j$ and $\mathfrak{p}_k$. It follows that if $\mathfrak{k}$ is a maximal $\mathrm{Ad}(K)$-invariant subalgebra of $\mathfrak{g}$, then A_3 and A_4 are both negative. On the other hand, if $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$ is an $\mathrm{Ad}(K)$-invariant proper intermediate subalgebra, then we may assume after reindexing that $A_4 = 0$. Furthermore, by the discussion on p. 182 of [10], for $i = 1, 2$, the constants $A_i = 0$ iff the identity component of K acts trivially on $\mathfrak{p}_i$, $\mathfrak{p}_i$ is Abelian, and $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_j$ for $j \neq i$.

Consequently, the possible weight vectors $w^{(j)}$ are $(0, -1)$, $(-1, 0)$, $(1, -2)$, $(-2, 1)$ and the non-trivial situations to consider are when the set $\mathcal{W}$ of weight vectors is:

(i) $\{(0, -1), (-1, 0)\}$,
(ii) $\{(0, -1), (1, -2)\}$,
(iii) $\{(0, -1), (-1, 0), (1, -2)\}$, or
(iv) $\{(0, -1), (-1, 0), (1, -2), (-2, 1)\}$.

Cases (i) and (ii) were analysed in [6]. Situation (i) is the case of doubly-warped product metrics where the hypersurface is a product of isotropy irreducible spaces (or more generally Einstein spaces with non-zero Einstein constant). Case (ii) can occur if the hypersurface is a torus bundle over an Einstein base. In both (i) and (ii) we have $m = 0$ so there are no additional constraints in the equations.

In this paper we shall study Cases (iii) and (iv). Case (iv) is when the hypersurface G/K has two inequivalent summands in its isotropy representation and where $\mathfrak{k}$ is a maximal $\mathrm{Ad}(K)$-invariant subalgebra in $\mathfrak{g}$. A connected homogeneous space G/K satisfying the latter maximality condition is called a *primitive* homogeneous space and by the proof of Theorem 2.2 in [10], there always exists on it a G-invariant Einstein metric. But the maximality condition also means that the cohomogeneity one manifold $(G/K) \times I$ cannot be compactified by adding singular orbits.

Thus Case (iii) is the generic situation when there are two distinct summands. In this case, the principal orbit admits a G-invariant Einstein metric iff $A_2^2 + 4A_1A_3(2d_1 + d_2)(d_2/d_1^2) \geq 0$ (cf. [10]). If equality holds, there is only one solution of the homogeneous Einstein equation; otherwise, there are two solutions.

We will now specialise the discussion in Section 2 to Cases (iii) and (iv). However, we will take the 2×2 matrix D to be $\mathrm{diag}(-1, 1)$ instead of $\mathrm{diag}(1, -1)$. In Case (iii) we have

$m=1$ and $d_1 \neq 1$ (otherwise $A_1=0$ by the above discussion). Then

$$\hat{U} = \begin{pmatrix} d_1-1 & d_2 \\ d_1 & d_2-1 \\ d_1+1 & d_2-2 \end{pmatrix}, \qquad \nu = \begin{pmatrix} -1 & 2 \end{pmatrix},$$

$$\hat{U} J \hat{U}^{\mathrm{T}} = \begin{pmatrix} 1-\dfrac{1}{d_1} & 1 & 1+\dfrac{1}{d_1} \\ 1 & 1-\dfrac{1}{d_2} & 1-\dfrac{2}{d_2} \\ 1+\dfrac{1}{d_1} & 1-\dfrac{2}{d_2} & 1-\dfrac{1}{d_1}-\dfrac{4}{d_2} \end{pmatrix},$$

$$\Theta = \sqrt{\frac{d_1}{d_1-1}} \begin{pmatrix} 0 & \dfrac{d_1-1}{d_1} & 0 \\ \sqrt{\dfrac{n-1}{d_1 d_2}} & 1 & 0 \\ 2\sqrt{\dfrac{n-1}{d_1 d_2}} & \dfrac{d_1+1}{d_1} & 0 \end{pmatrix}.$$

Rescaling v_1 by $\sqrt{d_1/(d_1-1)}\sqrt{(n-1)/d_1 d_2}$ and v_2 by $\sqrt{d_1/(d_1-1)}$, the equations are now

$$z_1' = \frac{2(d_1-1)}{d_1} z_1 v_2, \tag{3.2}$$

$$z_2' = z_2(2v_1 + 2v_2), \tag{3.3}$$

$$z_3' = z_3\left(4v_1 + \frac{2(d_1+1)}{d_1} v_2\right), \tag{3.4}$$

$$v_1' = \frac{1-n}{(d_1-1)d_2}\,(z_2 + 2z_3), \tag{3.5}$$

$$v_2' = z_1 + \left(\frac{d_1}{d_1-1}\right) z_2 + \left(\frac{d_1+1}{d_1-1}\right) z_3. \tag{3.6}$$

The Hamiltonian constraint is

$$\frac{-d_2(d_1-1)}{n-1} v_1^2 + \left(\frac{d_1-1}{d_1}\right) v_2^2 - z_1 - z_2 - z_3 = 0, \tag{3.7}$$

and the additional constraint is

$$z_2^2 = \kappa_1 z_1 z_3, \tag{3.8}$$

where $\kappa_1 = A_2^2/A_1A_3$, which is negative because A_1, A_2 are positive and A_3 is negative.

In Case (iv) we have $m = 2$, and we will assume that $d_i \geq 2$. Then

$$\hat{U} = \begin{pmatrix} d_1 - 1 & d_2 \\ d_1 & d_2 - 1 \\ d_1 + 1 & d_2 - 2 \\ d_1 - 2 & d_2 + 1 \end{pmatrix}, \qquad \nu = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

$$\hat{U} J \hat{U}^{\mathrm{T}} = \begin{pmatrix} 1 - \frac{1}{d_1} & 1 & 1 + \frac{1}{d_1} & 1 - \frac{2}{d_1} \\ 1 & 1 - \frac{1}{d_2} & 1 - \frac{2}{d_2} & 1 + \frac{1}{d_2} \\ 1 + \frac{1}{d_1} & 1 - \frac{2}{d_2} & 1 - \frac{1}{d_1} - \frac{4}{d_2} & 1 + \frac{2}{d_1} + \frac{2}{d_2} \\ 1 - \frac{2}{d_1} & 1 + \frac{1}{d_2} & 1 + \frac{2}{d_1} + \frac{2}{d_2} & 1 - \frac{4}{d_1} - \frac{1}{d_2} \end{pmatrix},$$

$$\Theta = \sqrt{\frac{d_1}{d_1 - 1}} \begin{pmatrix} 0 & \frac{d_1 - 1}{d_1} & 0 & 0 \\ \sqrt{\frac{n-1}{d_1 d_2}} & 1 & 0 & 0 \\ 2\sqrt{\frac{n-1}{d_1 d_2}} & \frac{d_1 + 1}{d_1} & 0 & 0 \\ -\sqrt{\frac{n-1}{d_1 d_2}} & \frac{d_1 - 2}{d_1} & 0 & 0 \end{pmatrix}.$$

Rescaling v_i as above the equations become

$$z_1' = \frac{2(d_1 - 1)}{d_1} z_1 v_2, \tag{3.9}$$

$$z_2' = z_2(2v_1 + 2v_2), \tag{3.10}$$

$$z_3' = z_3 \left(4v_1 + \frac{2(d_1 + 1)}{d_1} v_2 \right), \tag{3.11}$$

$$z_4' = z_4 \left(-2v_1 + \frac{2(d_1 - 2)}{d_1} v_2 \right), \tag{3.12}$$

$$v_1' = \frac{1 - n}{d_2(d_1 - 1)} \left(z_2 + 2z_3 - z_4 \right), \tag{3.13}$$

$$v_2' = z_1 + \left(\frac{d_1}{d_1 - 1} \right) z_2 + \left(\frac{d_1 + 1}{d_1 - 1} \right) z_3 + \left(\frac{d_1 - 2}{d_1 - 1} \right) z_4. \tag{3.14}$$

The Hamiltonian constraint is

$$-\frac{d_2(d_1-1)}{n-1}v_1^2+\left(\frac{d_1-1}{d_1}\right)v_2^2-z_1-z_2-z_3-z_4=0, \tag{3.15}$$

and the additional constraints are

$$z_2^2=\kappa_1 z_1 z_3, \tag{3.16}$$

$$z_1^2=\kappa_2 z_2 z_4, \tag{3.17}$$

where $\kappa_1=A_2^2/A_1A_3$ and $\kappa_2=A_1^2/A_2A_4$.

Remark 3.1. Our equations have polynomial right-hand side with only quadratic non-linearities. A majorisation argument (for example along the lines of that in [6]) shows that formal series solutions to the equations around a singularity will in fact converge on a punctured disc around the singularity.

4. Leading terms

Let us consider the generic case (iii). We shall first find the possible leading terms of a Painlevé expansion. We put

$$z_i=\alpha^{(i)}s^{m_i}+\cdots, \qquad v_i=\beta^{(i)}s^{n_i}+\cdots.$$

(a) First suppose that $m_1=0$. Now (3.2) implies that $n_2>-1$.
If $m_2\neq 0$ then $n_1=-1$ from (3.3), so equations (3.2)–(3.4) show that $(m_1,m_2,m_3)=\beta^{(1)}(0,2,4)$. Now (3.5) forces $\beta^{(1)}<0$ so m_3 is the least m_i, and now (3.5) and (3.6) show that $n_1=n_2$, a contradiction.
If $m_2=0$ then (3.2)–(3.4) imply $n_1,n_2>-1$ and hence $m_3=0$ also, and we have no singularity.

(b) We can therefore assume that $m_1\neq 0$ and hence $n_2=-1\leq n_1$ from (3.2)–(3.4).
If $n_1>-1$ then $(m_1,m_2,m_3)=2\beta^{(2)}((d_1-1)/d_1,1,(d_1+1)/d_1)$. If $\beta^{(2)}$ is positive then all m_i are positive and (3.5) gives a contradiction. If $\beta^{(2)}$ is negative then m_3 is the least m_i and so (3.5) and (3.6) force $n_1=n_2$, a contradiction.

(c) We now consider the case $n_1=n_2=-1$. Eqs. (3.2)–(3.4) show that

$$\begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix}=\begin{pmatrix} \dfrac{2(d_1-1)}{d_1}\beta^{(2)}\\ 2\beta^{(1)}+2\beta^{(2)}\\ 4\beta^{(1)}+\dfrac{2(d_1+1)}{d_1}\beta^{(2)}\end{pmatrix}.$$

Note that all the m_i are distinct unless $\beta^{(2)}+d_1\beta^{(1)}=0$, when they are all equal.
If m_1 is the least of the m_i, (3.5) and (3.6) shows that $n_2<n_1$, an immediate contradiction.

If m_2 is least, then (3.5) and (3.6) imply

$$(\beta^{(1)}, \beta^{(2)}) = \frac{\alpha^{(2)}}{d_1 - 1}\left(\frac{n-1}{d_2}, -d_1\right).$$

Substituting this into our expressions for m_i we find after some simplifying that

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = -2\alpha^{(2)} \begin{pmatrix} 1 \\ 1 - \dfrac{1}{d_2} \\ 1 - \dfrac{2}{d_2} \end{pmatrix},$$

contradicting the assumption that m_2 is least.

If m_3 is least then, as above, we use (3.5) and (3.6) to express $\beta^{(i)}$ in terms of $\alpha^{(i)}$:

$$(\beta^{(1)}, \beta^{(2)}) = \frac{\alpha^{(3)}}{d_1 - 1}\left(\frac{2(n-1)}{d_2}, -(d_1+1)\right).$$

We obtain the following expression for m_i:

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = -2\alpha^{(3)} \begin{pmatrix} 1 + \dfrac{1}{d_1} \\ 1 - \dfrac{2}{d_2} \\ 1 - \dfrac{1}{d_1} - \dfrac{4}{d_2} \end{pmatrix}.$$

As we are taking $m_3 < m_1, m_2$ we need $\alpha^{(3)} < 0$. Eqs. (3.5) and (3.6) imply that $m_3 = -2$, so

$$\alpha^{(3)} = \frac{1}{1 - (1/d_1) - (4/d_2)},$$

and

$$d_2 < \frac{4d_1}{d_1 - 1}.$$

(d) The last case to consider is when $\beta^{(2)} + d_1\beta^{(1)} = 0$ and hence all the m_i are equal (in fact they equal $2\beta^{(1)}(1 - d_1)$).

There are two possibilities from (3.5) and (3.6).

Either $m_i = -2$ and

$$\beta^{(1)} = \frac{n-1}{(d_1-1)d_2}(\alpha^{(2)} + 2\alpha^{(3)}), \quad \beta^{(2)} = -\alpha^{(1)} - \left(\frac{d_1}{d_1-1}\right)\alpha^{(2)} - \left(\frac{d_1+1}{d_1-1}\right)\alpha^{(3)},$$

or $m_i < -2$ and the linear combinations of $\alpha^{(i)}$ in the preceding equations are zero.

In the latter case we find that $\alpha^{(1)} = \alpha^{(3)} = -(1/2)\alpha^{(2)}$ and the constraint now gives a contradiction.

In the former case we obtain, using the equations $m_i = -2$ and $\beta^{(2)} + d_1\beta^{(1)} = 0$,

$$\beta^{(1)} = \frac{1}{d_1 - 1}, \qquad \beta^{(2)} = -\frac{d_1}{d_1 - 1}, \qquad \alpha^{(1)} = \alpha^{(3)} + \frac{d_1}{n-1},$$
$$\alpha^{(2)} = -2\alpha^{(3)} + \frac{d_2}{n-1}.$$

Imposing the constraint (3.8) now gives the equation

$$(\kappa_1 - 4)(\alpha^{(3)})^2 + \left(\frac{\kappa_1 d_1 + 4d_2}{n-1}\right)\alpha^{(3)} - \left(\frac{d_2}{n-1}\right)^2 = 0. \tag{4.1}$$

Recall that κ_1 is negative, in particular does not equal to 4, and so we have a genuine quadratic equation for $\alpha^{(3)}$. Let us now introduce a variable τ by the relation

$$\left(2 + \frac{A_2}{A_1}\tau\right)((n-1)\alpha^{(3)} + d_1) = 2d_1 + d_2.$$

Indeed, τ is just the asymptotic value of f_1^2/f_2^2 (cf. (9.1) and (9.2)), which is given by $A_1\alpha^{(2)}/A_2\alpha^{(1)}$. Under the transformation from $\alpha^{(3)}$ to τ, the quadratic equation (4.1) becomes a quadratic equation in τ which is precisely the Einstein condition for G-invariant metrics on the principal orbit G/K.

We summarise our results in the following theorem.

Theorem 4.1. *The possible leading terms for Case* (iii) *are as follows*:

(I) *If* $d_2 < 4d_1/(d_1 - 1)$ *we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \frac{2d_1 d_2}{4d_1 - d_2(d_1 - 1)} \begin{pmatrix} 1 + \dfrac{1}{d_1} \\ 1 - \dfrac{2}{d_2} \\ \dfrac{4d_1 - d_2(d_1-1)}{-d_1 d_2} \end{pmatrix}, \qquad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \dfrac{-d_1 d_2}{4d_1 - d_2(d_1 - 1)} \\ \dfrac{2(n-1)}{d_2(d_1-1)}\alpha^{(3)} \\ -\left(\dfrac{d_1+1}{d_1-1}\right)\alpha^{(3)} \end{pmatrix},$$

where the constraint (3.8) becomes the relation

$$(\alpha^{(2)})^2 = \frac{-\kappa_1 d_1 d_2}{4d_1 - d_2(d_1-1)}\alpha^{(1)}.$$

(II) *In all cases we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} \alpha^{(3)} + \dfrac{d_1}{n-1} \\ -2\alpha^{(3)} + \dfrac{d_2}{n-1} \\ \alpha^{(3)} \\ \dfrac{1}{d_1-1} \\ \dfrac{-d_1}{d_1-1} \end{pmatrix},$$

where $\alpha^{(3)}$ *is a root of*

$$(\kappa_1 - 4)x^2 + \left(\frac{\kappa_1 d_1 + 4d_2}{n-1}\right)x - \left(\frac{d_2}{n-1}\right)^2 = 0.$$

Each real root $\alpha^{(3)}$*, necessarily negative, corresponds to a* G*-invariant Einstein metric on* G/K *of volume* 1.

Remark 4.2. It is interesting to observe that if the inequality in the condition for Case (I) to arise is replaced by the *equality*

$$d_2 = \frac{4d_1}{d_1-1},$$

we obtain the condition under which conserved quantities were found for double warped product metrics in [4]. These warped product metrics correspond of course to Case (i) of Section 3, rather than Case (iii) which we are analysing here.

Remark 4.3. The leading terms in (II) are expected by a priori consideration when G/K admits a G-invariant Einstein metric with positive constant because the metric cone over it is Ricci-flat. Conversely, the asymptotic behaviour discussed in Section 9 associates the leading terms with a G-invariant Einstein metric on G/K. Note also that in (I) we have $m_1, m_2 \geq 0$ so only z_3, v_1, v_2 blow up, whereas in (II) all the variables blow up. See Section 9 for more discussion of the asymptotic geometric behaviour of the leading terms.

5. Resonances

The next step is to compute the resonances for each set of possible leading terms.

We substitute

$$z_i = \sum_{j=0}^{\infty} \alpha_j^{(i)} s^{m_i+(j/Q)}, \qquad v_i = \sum_{j=0}^{\infty} \beta_j^{(i)} s^{-1+(j/Q)},$$

where Q is some integer to be determined later, into the equations. In the notation of the previous section, $\alpha^{(i)} = \alpha_0^{(i)}$, $\beta^{(i)} = \beta_0^{(i)}$. Equating powers, we obtain the following recursion relations (valid for $j \neq 0$):

- Case (I)

$$\begin{pmatrix} \frac{j}{Q} & 0 & 0 & 0 & \frac{-2(d_1-1)}{d_1}\alpha_0^{(1)} \\ 0 & \frac{j}{Q} & 0 & -2\alpha_0^{(2)} & -2\alpha_0^{(2)} \\ 0 & 0 & \frac{j}{Q} & -4\alpha_0^{(3)} & -\frac{2(d_1+1)}{d_1}\alpha_0^{(3)} \\ 0 & 0 & \frac{2(n-1)}{(d_1-1)d_2} & \frac{j}{Q}-1 & 0 \\ 0 & 0 & -\left(\frac{d_1+1}{d_1-1}\right) & 0 & \frac{j}{Q}-1 \end{pmatrix} \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \\ \alpha_j^{(3)} \\ \beta_j^{(1)} \\ \beta_j^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2(d_1-1)}{d_1}\sum_{i=1}^{j-1} \alpha_i^{(1)}\beta_{j-i}^{(2)} \\ 2\sum_{i=1}^{j-1} \alpha_i^{(2)}(\beta_{j-i}^{(1)} + \beta_{j-i}^{(2)}) \\ \sum_{i=1}^{j-1} \alpha_i^{(3)}\left(4\beta_{j-i}^{(1)} + \frac{2(d_1+1)}{d_1}\beta_{j-i}^{(2)}\right) \\ \frac{1-n}{(d_1-1)d_2}\alpha_{j-Q(m_2+2)}^{(2)} \\ \alpha_{j-Q(m_1+2)}^{(1)} + \frac{d_1}{d_1-1}\alpha_{j-Q(m_2+2)}^{(2)} \end{pmatrix}. \tag{5.1}$$

- Case (II)

$$\begin{pmatrix} \frac{j}{Q} & 0 & 0 & 0 & -\frac{2(d_1-1)}{d_1}\alpha_0^{(1)} \\ 0 & \frac{j}{Q} & 0 & -2\alpha_0^{(2)} & -2\alpha_0^{(2)} \\ 0 & 0 & \frac{j}{Q} & -4\alpha_0^{(3)} & -\frac{2(d_1+1)}{d_1}\alpha_0^{(3)} \\ 0 & \frac{n-1}{(d_1-1)d_2} & \frac{2(n-1)}{(d_1-1)d_2} & \frac{j}{Q}-1 & 0 \\ -1 & \frac{-d_1}{d_1-1} & -\left(\frac{d_1+1}{d_1-1}\right) & 0 & \frac{j}{Q}-1 \end{pmatrix} \begin{pmatrix} \alpha_j^{(1)} \\ \alpha_j^{(2)} \\ \alpha_j^{(3)} \\ \beta_j^{(1)} \\ \beta_j^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2(d_1-1)}{d_1}\displaystyle\sum_{i=1}^{j-1}\alpha_i^{(1)}\beta_{j-i}^{(2)} \\ 2\sum_{i=1}^{j-1}\alpha_i^{(2)}(\beta_{j-i}^{(1)}+\beta_{j-i}^{(2)}) \\ \sum_{i=1}^{j-1}\alpha_i^{(3)}\left(4\beta_{j-i}^{(1)}+\dfrac{2(d_1+1)}{d_1}\beta_{j-i}^{(2)}\right) \\ 0 \\ 0 \end{pmatrix}. \tag{5.2}$$

Notice that $Q(m_1+2)=2Q(m_2+2)$ in Case (I).

In the above recursions we set $R=j/Q$ and denote the coefficient matrix on the left-hand side by $Z(R)$. The resonances correspond to the steps of the recursion at which free parameters enter. More precisely, they are (for $R\neq 0$) the values of $R=j/Q$ for which $Z(R)$ is non-invertible. To compute the resonances, let us therefore replace the right-hand side of the recursion by zero and see when there is a non-trivial solution to the equation $Z(R)\mathbf{x}=0$.

In Case (I) we can use the first three equations to eliminate $\alpha_j^{(i)}$ and we are left with

$$R(R-1)\begin{pmatrix}\beta_j^{(1)}\\ \beta_j^{(2)}\end{pmatrix}=\alpha_0^{(3)}\begin{pmatrix}\dfrac{8(1-n)}{d_2(d_1-1)} & \dfrac{4(1-n)(d_1+1)}{d_1d_2(d_1-1)}\\ \dfrac{4(d_1+1)}{d_1-1} & \dfrac{2(d_1+1)^2}{d_1(d_1-1)}\end{pmatrix}\begin{pmatrix}\beta_j^{(1)}\\ \beta_j^{(2)}\end{pmatrix}. \tag{5.3}$$

The matrix on the right (including the $\alpha_0^{(3)}$ factor) clearly has determinant zero. Moreover, after some calculation we find its trace to be

$$2\left(1-\frac{1}{d_1}-\frac{4}{d_2}\right)\alpha_0^{(3)}=2, \tag{5.4}$$

so the matrix has eigenvalues 0 and 2. The resonances are therefore the roots of $R(R-1)=0$ or $R(R-1)=2$, that is $-1, 0, 1, 2$. (Eq. (5.1) is not valid for $R=0$ but zero is still a resonance because the leading terms contain a free parameter.)

In Case (II) we proceed similarly and find that the resonances are given by the roots of $R(R-1)=\lambda$, where λ is an eigenvalue of

$$\begin{pmatrix} \dfrac{1-n}{d_2(d_1-1)}(2\alpha_0^{(2)}+8\alpha_0^{(3)}) & \dfrac{1-n}{d_2(d_1-1)}\left(2\alpha_0^{(2)}+4\left(\dfrac{d_1+1}{d_1}\right)\alpha_0^{(3)}\right)\\ \dfrac{2}{d_1-1}(d_1\alpha_0^{(2)}+2(d_1+1)\alpha_0^{(3)}) & 2\left(\dfrac{d_1-1}{d_1}\right)\alpha_0^{(1)}+\left(\dfrac{2d_1}{d_1-1}\right)\alpha_0^{(2)}+\dfrac{2(d_1+1)^2}{d_1(d_1-1)}\alpha_0^{(3)} \end{pmatrix}.$$

Using the formulae of Theorem 4.1 for $\alpha_0^{(i)}$ we can simplify this matrix to

$$\begin{pmatrix} \dfrac{2d_1}{1-d_1}+2+\dfrac{4(n-1)}{d_2(1-d_1)}\alpha_0^{(3)} & \dfrac{2}{1-d_1}+\dfrac{4(n-1)}{d_1d_2(1-d_1)}\alpha_0^{(3)}\\ \dfrac{2}{1-d_1}\left(\dfrac{d_1d_2}{1-n}\right)+\dfrac{4}{d_1-1}\alpha_0^{(3)} & \dfrac{2d_2}{(1-d_1)(1-n)}+2+\dfrac{4}{d_1(d_1-1)}\alpha_0^{(3)} \end{pmatrix}.$$

Clearly, 2 is an eigenvalue of this matrix, while from computing the trace we see the other eigenvalue is $2/(1-n) - (4n/d_1d_2)\alpha_0^{(3)}$.

Theorem 5.1. *The resonances in the Painlevé analysis of the system (3.2)–(3.6) with constraint (3.8) are as follows*:

Case (I) $R = -1, 0, 1, 2,$
Case (II) $R = -1, 2$ *and the* (*non-zero*) *roots of*

$$R(R-1) = \frac{2}{1-n} - \frac{4n}{d_1d_2}\alpha_0^{(3)},$$

where $\alpha_0^{(3)}$ *is given by* (4.1).

Remark 5.2. The appearance of $R = -1$ as a resonance is typical for autonomous systems of differential equations, and is associated to the degree of freedom we have in translating the independent variable s. We shall see in Section 6 that the degree of freedom from the resonance at $R = 2$ is fixed by the Hamiltonian constraint.

Remark 5.3. In Case (II), a *necessary* condition for there to be a rational resonance other than $-1, 2$ is that $\alpha_0^{(3)}$ is rational. From (4.1) this condition is

$$\sqrt{\kappa_1^2 d_1^2 + 8\kappa_1 d_1 d_2 + 4\kappa_1 d_2^2} \in \mathbb{Q}. \tag{5.5}$$

Remark 5.4. Denoting $2/(1-n) - (4n/d_1d_2)\alpha_0^{(3)}$ by θ, we have the following table relating the value of θ to the values of the roots $R_1 \leq R_2$ of $R(R-1) = \theta$. Of course $R_1 + R_2 = 1$:

$$\begin{aligned}
&\theta < -\tfrac{1}{4}, \quad R_1, R_2 \in \mathbb{C} - \mathbb{R}, \qquad \theta = -\tfrac{1}{4}, \quad R_1 = R_2 = \tfrac{1}{2},\\
&-\tfrac{1}{4} < \theta < 0, \quad 0 < R_1 < R_2 < 1, \qquad \theta = 0, \quad R_1 = 0, R_1 = 1,\\
&0 < \theta < 2, \quad -1 < R_1 < 0 < 1 < R_2 < 2, \qquad \theta = 2, \quad R_1 = -1, R_2 = 2,\\
&\theta > 2, \quad R_1 < -1 < 2 < R_2.
\end{aligned}$$

Remark 5.5. One can easily check that $\theta = 0$ if and only if the quadratic equation for $\alpha_0^{(3)}$ has a repeated root (that is, if and only if G/K admits a *unique* G-invariant Einstein metric). In this situation the Painlevé expansion for Case (II) has resonances $-1, 1, 2$.

6. Compatibility conditions

At each resonance we must check that the recursion is solvable. Let us first consider Case (I).

Observe first from Theorem 4.1 that m_1 is positive, m_2 is non-negative and m_2 vanishes if and only if $d_1 = 2$. For the steps in the recursion up to and including the top resonance

$R = 2$, it follows that, except when $d_1 = 2$, the terms in the last two entries of the right-hand side of (5.1) are zero.

For $j < Q$, since Z is invertible, we see that $\alpha_j^{(i)}$, $\beta_j^{(i)}$ and the right-hand side of (5.1), are all zero. At $R = 1$, that is $j = Q$, solving the recursion is just finding the kernel of Z. So we have

$$\begin{pmatrix} \alpha_Q^{(1)} \\ \alpha_Q^{(2)} \\ \alpha_Q^{(3)} \\ \beta_Q^{(1)} \\ \beta_Q^{(2)} \end{pmatrix} = \mu \begin{pmatrix} 4(1-d_1)\alpha_0^{(1)} \\ 2(1-d_1)\alpha_0^{(2)} \\ 0 \\ d_1+1 \\ -2d_1 \end{pmatrix}, \tag{6.1}$$

where μ is a free parameter.

At $R = 2$, that is $j = 2Q$, the kernel of Z^{T} is spanned by $(0, 0, 1, 4\alpha_0^{(3)}, (2(d_1 + 1)/d_1)\alpha_0^{(3)})^{\mathrm{T}}$ so if $d_1 \neq 2$ the compatibility condition becomes

$$\sum_{i=1}^{2Q-1} \alpha_i^{(3)} \left(4\beta_{2Q-i}^{(1)} + \frac{2(d_1+1)}{d_1}\beta_{2Q-i}^{(2)} \right) = 0. \tag{6.2}$$

Note that this is true even if $d_1 = 2$ because in this case the terms in the last two places on the right-hand side of (5.1) are $\sigma_1 = [(1-n)/(d_1-1)d_2]\alpha_0^{(2)}$ and $\sigma_2 = (d_1/(d_1-1))\alpha_0^{(2)}$, respectively, and it is easy to check that if $d_1 = 2$ then $4\sigma_1 + (2(d_1 + 1)/d_1)$ $\sigma_2 = 0$.

Now we have already observed that for $0 < j < Q$ the quantities $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are zero. It follows that the only term which contributes to the sum in (6.2) is the one with $i = Q$, and this is zero because from (6.1) $\alpha_Q^{(3)}$ is zero. Hence the recursion is solvable at $j = 2Q$ and we are done.

The free parameter entering the Painlevé expansion at the top resonance is just the freedom to add an element of ker $Z(2)$ to $(\alpha_{2Q}^{(1)}, \dots, \beta_{2Q}^{(2)})$. We can take

$$\left(\left(\frac{1-d_1}{d_1}\right)\alpha_0^{(1)}, \left(\frac{2(n-1)}{(d_1+1)d_2} - 1\right)\alpha_0^{(2)}, \left(\frac{4(n-1)}{d_2(d_1+1)} - \frac{d_1+1}{d_1}\right)\alpha_0^{(3)}, \right.$$
$$\left. \frac{2(n-1)}{(d_1+1)d_2}, -1\right)^{\mathrm{T}}$$

as a generator of the kernel. Since the Hamiltonian is constant along a solution of Eqs. (3.2)–(3.6), the value of the constant in the present situation is determined by the constant term of the expansion of the Hamiltonian in powers of s, and this equals

$$\left(0, 0, -1, -4\alpha_0^{(3)}, -\frac{2(d_1+1)}{d_1}\alpha_0^{(3)}\right) \cdot (\alpha_{2Q}^{(1)}, \alpha_{2Q}^{(2)}, \alpha_{2Q}^{(3)}, \beta_{2Q}^{(1)}, \beta_{2Q}^{(2)}) \tag{6.3}$$

plus terms with subscripts $j < 2Q$. The scalar product of the left-hand vector of (6.3) with our generator of the kernel of $Z(2)$ is

$$3\alpha_0^{(3)}\left(\frac{d_1+1}{d_1}-\frac{4(n-1)}{(d_1+1)d_2}\right)=\frac{3\alpha_0^{(3)}(d_1-1)}{d_1d_2(d_1+1)}(d_2(d_1-1)-4d_1)\neq 0$$

by assumption, so the free parameter at the top resonance is fixed by the Hamiltonian constraint. Finally, we observe that by Remark 2.1, z_2^2/z_1z_3 is a conserved quantity of our equations, and that its value on our Painlevé series solution is determined by the leading terms. Since we have already chosen the coefficients in the leading terms to satisfy the constraint, it follows that (3.8) holds. We have now proved the following theorem.

Theorem 6.1. *If* $d_2 < 4d_1/(d_1-1)$ *then family* (I) *gives rise to a convergent Painlevé expansion satisfying all the constraints and depending on the full number of parameters.*

Remark 6.2. One can check that for all the values of d_1, d_2 satisfying the inequality of Theorem 6.1, except the case $d_2 = 3$, $d_1 \neq 3$, we have m_i integral. As the resonances R are always integral, we can therefore take $Q = 1$ except in this special case. So our expansions are actually meromorphic in s (rather than a fractional power of s) unless $d_2 = 3$, $d_1 \neq 3$.

For family (II) we can prove the following result at the top resonance.

Lemma 6.3. *The recursion in family* (II) *at the top resonance is solvable provided the earlier recursions are solvable. Moreover, the free parameter at the top resonance is fixed by the Hamiltonian constraint.*

Proof. One can easily verify that the kernel of $Z(2)^{\mathrm{T}}$ is spanned by $(1, 1, 1, 2d_2/(n-1), 2)^{\mathrm{T}}$. The compatibility condition at the top resonance $R = 2$ (that is, $j = 2Q$) is therefore

$$\frac{2(d_1-1)}{d_1}\sum_{i=1}^{2Q-1}\alpha_i^{(1)}\beta_{2Q-i}^{(2)}+2\sum_{i=1}^{2Q-1}\alpha_i^{(2)}(\beta_{2Q-i}^{(1)}+\beta_{2Q-i}^{(2)})$$
$$+\sum_{i=1}^{2Q-1}\alpha_i^{(3)}\left(4\beta_{2Q-i}^{(1)}+\frac{2(d_1+1)}{d_1}\beta_{2Q-i}^{(2)}\right)=0.$$

We can write this as

$$2\sum_{i=1}^{2Q-1}(\alpha_i^{(2)}+2\alpha_i^{(3)})\beta_{2Q-i}^{(1)}$$
$$+\frac{2(d_1-1)}{d_1}\sum_{i=1}^{2Q-1}\left(\alpha_i^{(1)}+\frac{d_1}{d_1-1}\alpha_i^{(2)}+\frac{d_1+1}{d_1-1}\alpha_i^{(3)}\right)\beta_{2Q-i}^{(2)}=0,$$

and hence, using the fourth and fifth rows of the recursion (5.2), as

$$\frac{2(d_1-1)d_2}{1-n}\sum_{i=1}^{2Q-1}\left(\frac{i}{Q}-1\right)\beta_i^{(1)}\beta_{2Q-i}^{(1)}+\frac{2(d_1-1)}{d_1}\sum_{i=1}^{2Q-1}\left(\frac{i}{Q}-1\right)\beta_i^{(2)}\beta_{2Q-i}^{(2)}=0.$$

Changing the index to $k = 2Q - i$ we see that both terms on the left-hand side now vanish.

The kernel of $Z(2)$ is generated by $((1-d_1)\alpha_0^{(1)}, (1-d_1)\alpha_0^{(2)}, (1-d_1)\alpha_0^{(3)}, 1, -d_1)$. On the other hand, the constant term in the Hamiltonian is

$$\left(-1,-1,-1,\frac{-2d_2}{n-1},-2\right)\cdot(\alpha_{2Q}^{(1)},\alpha_{2Q}^{(2)},\alpha_{2Q}^{(3)},\beta_{2Q}^{(1)},\beta_{2Q}^{(2)})$$

plus terms with subscripts $j < 2Q$. The inner product of the vector on the left with the generator of $\ker(Z(2))$ is $3(d_1-1)n/(n-1) \neq 0$, proving our last claim. □

Remark 6.4. We can also observe that in Case (II) compatibility always holds at the step j_1 corresponding to the first positive resonance, because $\alpha_j^{(i)}, \beta_j^{(i)}$ and the right-hand side of (5.2) are zero for $j < j_1$.

However, if the resonances R_1, R_2 other than $-1, 2$ satisfy $0 < R_1 < R_2 < 2$ it may happen that the compatibility condition at R_2 is *not* satisfied. In Section 7 we shall see an example ($G/K = \mathrm{SO}(5)/U(2)$) of this kind. In this example the compatibility condition at R_2 only holds if the free parameter at R_1 is set to zero, so we do not obtain a Painlevé expansion depending on the full number of parameters.

The following result gives a sufficient condition for all compatibility conditions to hold.

Lemma 6.5. *If*

$$-\frac{1}{4}<\frac{2}{1-n}-\frac{4n}{d_1d_2}\alpha_0^{(3)}<-\frac{2}{9}$$

then all compatibility conditions hold and we have a Painlevé expansion of type (II) *depending on the full number of parameters*.

Proof. We observed above that $\alpha_j^{(i)}, \beta_j^{(i)}$ are zero for $j < j_1$. It follows that the right-hand side of (5.2) is still zero for $j < 2j_1$, so compatibility in fact holds for $j < 2j_1$. So if we have $0 < R_1 < R_2 < \min(2R_1, 2)$ then all compatibility conditions hold. In the notation of Remark 5.4, we see that R_1, R_2 satisfy these inequalities if and only if $-(1/4) < \theta < 2/9$. □

Remark 6.6. As remarked earlier, $\alpha_0^{(3)}$ is negative when real. It follows that a *necessary* (but not sufficient) condition for the hypothesis of Lemma 6.5 to hold is that $n < 10$. In Section 7 we shall see an example ($G/K = \mathrm{Sp}(2)\mathrm{Sp}(1)/\mathrm{Sp}(1)\mathrm{Sp}(1)$ and $n = 7$), where Lemma 6.5 applies. On the other hand, the example $G/K = \mathrm{SO}(5)/U(2)$ below shows that $n < 10$ is not *sufficient* for either the hypothesis or the conclusion of Lemma 6.5 to hold.

7. Examples

Example 7.1 ($G/K = \mathrm{SO}(5)/U(2) \approx \mathbb{CP}^3$). This is the total space of the twistor space of $S^4 \approx \mathrm{SO}(5)/\mathrm{SO}(4)$ (as a self-dual manifold), so $d_1 = 2$ and $d_2 = 4$. The constant κ_1 is -36. Note that $\mathrm{SO}(5)/U(2)$ admits exactly two SO(5)-invariant Einstein metrics: the Fubini–Study metric and the Einstein metric induced by the Killing form of SO(5), which was first found in [11].

As $d_2 < 4d_1/(d_1 - 1)$ the expansion (I) exists, with $(m_1, m_2, m_3) = (6, 2, -2)$.

For (II) we have: (a) $\alpha_0^{(3)} = -(1/5)$ or (b) $\alpha_0^{(3)} = -(2/25)$. By Theorem 5.1 the resonances are:

(IIa) $R = -1, 2$ and the two irrational roots of $R(R-1) = (1/5)$;
(IIb) $R = -1, 1/5, 4/5, 2$.

After imposing the Hamiltonian constraint, the Painlevé expansion from (IIa) depends only on one free parameter, and represents, up to translation of s, the cone over the Fubini–Study metric.

For (IIb), we obtain a Painlevé expansion in powers of $s^{1/5}$. The compatibility condition at $R = 1/5$ holds automatically, but computations by hand or MAPLE show that compatibility at $R = 4/5$ forces the free parameter at $R = 1/5$ to be zero. The upshot is that the Ricci-flat equations admit a 2-parameter Painlevé expansion, where one free parameter is translation of s and the other comes from the resonance 4/5.

One can check that this family in fact represents the metrics of G_2 holonomy found in [3,7]. To do this, we first show that the condition for a cohomogeneity one G_2-metric is given by

$$2v_1^2 = -25z_3, \qquad \left(\frac{6v_1}{5} + v_2\right)^2 = 2z_1,$$

and the constraint equation $z_2^2 = -36z_1z_3$. These equations cut out a surface in the space of the z_i and v_j. One can then check that the Hamiltonian vector field is tangent to this surface. Using v_1 and v_2 to parameterise this surface, we obtain the following quadratic subsystem of (3.2)–(3.6):

$$v_1' = v_1(2v_1 + \tfrac{3}{2}v_2), \qquad v_2' = -\tfrac{6}{5}v_1(2v_1 + v_2) + \tfrac{1}{2}v_2^2.$$

Next, we show that this subsystem has a 2-parameter family of Painlevé expansions with leading terms for v_j exactly the same those for v_j in (IIb). Furthermore, the resonances are now at -1 and 4/5. Expressing the z_i in terms of v_j we recover the 2-parameter family of Painlevé expansions we obtained in (IIb), and so they indeed come from G_2-metrics.

This orbit type has a higher dimensional generalisation as follows.

Example 7.2 ($G/K = \mathrm{Sp}(m+1)/(\mathrm{Sp}(m)U(1)) \approx \mathbb{CP}^{2m+1}$, the twistor space of $\mathbb{HP}^m$). Now $d_1 = 2$, $d_2 = 4m$ and $\kappa_1 = -4m(m+2)^2$. There are again two invariant Einstein metrics on G/K, the Fubini–Study metric and the Ziller metric, neither of which are induced by the Killing form when $m > 1$.

For $m = 1$ this is just the previous example.

For $m > 1$ the inequality $d_2 < 4d_1/(d_1 - 1)$ no longer holds, so expansions of type (I) no longer exist. (Note that equality holds precisely when $m = 2$.)

For family (II) we find that the possibilities for $\alpha_0^{(3)}$ are:

(a) $\alpha_0^{(3)} = -(2m/(m+1)(4m+1))$, or
(b) $\alpha_0^{(3)} = -(2m/(m^2+3m+1)(4m+1))$,

corresponding to the Fubini–Study metric on G/K and the Ziller Einstein metric, respectively.

We find from Theorem 5.1 that the conditions for there to be rational resonances other than $-1, 2$ are, respectively:

(IIa) $(4m^2 + 13m + 1)(m + 1)(4m + 1)$ is a perfect square,
(IIb) $(4m^3 + 5m^2 - m + 1)(m^2 + 3m + 1)(4m + 1)$ is a perfect square.

For (IIa), let us write $\phi(m), \rho(m), \psi(m)$ for the factors $4m^2 + 13m + 1, m + 1, 4m + 1$, respectively ($m > 1$). It is straightforward to show by repeated long division of polynomials that $\phi(m), \psi(m)$ are always coprime, while the only common prime factors of $\phi(m), \rho(m)$ (respectively, $\rho(m), \psi(m)$) are 2 (respectively, 3). If $\phi(m)\rho(m)\psi(m)$ is a perfect square, therefore, we can write

$$\phi(m) = 2^j p_1^{2N_1} \cdots p_a^{2N_a}, \qquad \rho(m) = 2^k 3^\ell p_{a+1}^{2N_{a+1}} \cdots p_{a+b}^{2N_{a+b}},$$
$$\psi(m) = 3^q p_{a+b+1}^{2N_{a+b+1}} \cdots p_{a+b+c}^{2N_{a+b+c}},$$

where $j + k$ and $q + \ell$ are even.

Now $(2m + 2)^2 < \phi(m) < (2m + 4)^2$, and we see that $\phi(m)$ is a square only if $m = 8$ (when $\phi(m) = (2m + 3)^2$). In this case $\phi(m)\rho(m)\psi(m)$ is not a perfect square, so we may assume from now on that j and hence k is odd. Moreover, $3\psi(m) \equiv 3$ modulo 4 so is not a square, so we deduce that q and hence ℓ are even. It follows that $\psi(m) = 4m + 1$ and $2\rho(m) = 2m + 2$ are squares. Working modulo 8, and noting that squares are congruent to 0, 1 or 4 modulo 8, we see that this is impossible.

For (IIb) similar arguments show that

$$4m^3 + 5m^2 - m + 1, \qquad \tfrac{1}{5}(4m + 1), \qquad \tfrac{1}{5}(m^2 + 3m + 1)$$

must all be squares. In particular, we must have an integral point on the elliptic curve

$$y^2 = 4m^3 + 5m^2 - m + 1.$$

Siegel's theorem shows that there can only be finitely many such m. Moreover, the program Ratpoints of Elkies, Stahlke and Stoll shows that the only integral points (m, y) with $1 \le m \le 10^6$ are (1, 3) and (32, 369), and for the latter $(4m + 1)/5$ is not integral.

To summarise, if $m = 1$ (i.e. M has dimension 7) we have non-trivial Painlevé expansions of types (I) and (IIb). If $m > 1$ then there is no expansion of type (I) and, for the range of m that we have checked (up to 10^6) no type (IIb) expansion either (we conjecture that there is no (IIb) expansion for any $m > 1$). There is never a non-trivial expansion with leading term (IIa).

Example 7.3 ($G/K = G_2/(\mathrm{SU}(2)\mathrm{SO}(2))$). This is the twistor space of the quaternionic symmetric space $G_2/\mathrm{SO}(4)$. Now $d_1 = 2, d_2 = 8$ and $\kappa_1 = -128$. These are the same values as in case $m = 2$ of Example 7.2, so we deduce that there are no rational resonances other than $-1, 2$ and hence only a trivial Painlevé expansion.

Example 7.4 ($G/K = (\mathrm{Sp}(m+1)\mathrm{Sp}(1))/(\mathrm{Sp}(m)\Delta\mathrm{Sp}(1)) \approx S^{4m+3}$). Now $\mathrm{Sp}(m)\Delta\mathrm{Sp}(1) \subset \mathrm{Sp}(m)\mathrm{Sp}(1)\mathrm{Sp}(1)$ and so $d_1 = 3, d_2 = 4m$. Also we may compute $\kappa_1 = -(32/9)m(m+2)^2$.

Note that the inequality $d_2 < 4d_1/(d_1 - 1)$ needed for family (I) to exist is satisfied only for $m = 1$. In this case we have $(m_1, m_2, m_3) = (8, 3, -2)$ and Theorem 6.1 gives a full Painlevé family.

For family (II), we find from (4.1) that the two possible values for $\alpha_0^{(3)}$ are:

(a) $\alpha_0^{(3)} = -3m/(2m + 1)^2$, which corresponds to the constant curvature metric;
(b) $\alpha_0^{(3)} = -3m/(2m + 1)(4m^2 + 14m + 9)$, which corresponds to the Jensen squashed metric [8].

By Theorem 5.1 the resonances are:

(IIa) $R = -1, -1/(2m + 1), (2m + 2)/(2m + 1), 2$;
(IIb) $R = -1, 2$ and the roots of $R(R-1) = -(4m^2+10m+6)/(2m+1)(4m^2+14m+9)$.

In (IIa) since compatibility automatically holds at the first positive resonance, it follows from Lemma 6.3 we get a 2-parameter family of Painlevé expansions for the Ricci-flat equations.

In Case (IIb) we have four rational resonances if

$$(8m^3 + 16m^2 - 8m - 15)(2m + 1)(4m^2 + 14m + 9) \tag{7.1}$$

is a perfect square; otherwise $-1, 2$ are the only rational resonances, in which case by Lemma 6.3 the Painlevé expansion is just the cone over the Jensen metric, modulo the position of the singularity.

Using similar methods to those in Example 7.2, we see that if (7.1) is a square, then $8m^3 + 16m^2 - 8m - 15$, $(2m + 1)/3$ and $(4m^2 + 14m + 9)/3$ are all squares. In particular, the elliptic curve $y^2 = 8m^3 + 16m^2 - 8m - 15$ will have integral points. As before, Siegel's theorem tells us there are only finitely many such points. On the other hand, Ratpoints shows that the only integral points (m, y) with $1 \le m \le 10^6$ are (1, 1), (8, 71), (20, 265) and (68, 1609). The only one of these where $(2m + 1)/3$ equals a square is $m = 1$. Hence for $1 \le m \le 10^6$, only $m = 1$ satisfies our Diophantine condition.

In the special case $m = 1$ of (IIb), the resonances are $R = -1, 4/9, 5/9, 2$. Thus we can take $Q = 9$ so positive resonances occur at $j = 4, 5, 18$. Now Lemma 6.5 tells us that all the compatibility conditions hold. Hence we get a full 3-parameter Painlevé family for the Ricci-flat equations.

Using arguments similar to those at the end of Example 7.1, one can show that there is a 2-parameter subfamily of the above 3-parameter Painlevé family which corresponds to the (local) metrics with Spin(7) holonomy found in [3,7]. This 2-parameter subfamily is obtained by setting the free parameter at the first positive resonance to be 0. We note that the Spin(7) condition is given here by the equations

$$4v_1^2 = -27z_3, \qquad \left(\frac{8v_1}{3} + 2v_2\right)^2 = 6z_1,$$

and the constraint equation $z_2^2 = -32z_1z_3$.

To summarise, if $m = 1$ (i.e. M is eight-dimensional) we have non-trivial Painlevé expansions of types (I)–(IIb), with those of types (I) and (IIb) depending on the full number of parameters. If $m > 1$ there is no expansion of type (I) and, at least for $1 < m \leq 10^6$, no type (IIb) expansion either (we conjecture that there is no (IIb) expansion for any $m > 1$). Type (IIa) expansion exists for all m.

Example 7.5. Take $G/K = \mathrm{SO}(2m)/(\mathrm{SO}(m)U(1))$, where $\mathrm{SO}(m)U(1) \subset U(m) \subset \mathrm{SO}(2m)$ and $m \geq 3$. Since

$$d_1 = \tfrac{1}{2}(m+2)(m-1), \qquad d_2 = m(m-1),$$

we have $d_2 \geq 4d_1/(d_1 - 1)$, and so Painlevé expansions of type (I) never occur. From Wang and Ziller 10[10, p. 189] there is no G-invariant Einstein metric on G/K for $m > 3$, while for $m = 3$ there is a unique such metric. So for $m > 3$ we obtain examples of principal orbits for which the system (3.2)–(3.6) with constraints (3.7) and (3.8) has no (real) Painlevé expansions.

On the other hand, when $m = 3$, Eq. (4.1) has the unique root $\alpha_0^{(3)} = -(3/22)$, and as in Remark 5.5 the resonances are $-1, 1, 2$. We therefore obtain a 2-parameter Painlevé expansion of type (II) for the Ricci-flat equations.

8. The case of a primitive principal orbit

Let us now consider Case (iv) of Section 3, that is, when $\mathfrak{k}$ is a maximal $\mathrm{Ad}(K)$-invariant subalgebra of $\mathfrak{g}$. Recall that in Section 3 we have defined constants $\kappa_1 = A_2^2/A_1A_3$ and $\kappa_2 = A_1^2/A_2A_4$ which are negative.

Theorem 8.1. *The possible leading terms are as follows*:

(Ia) *If* $d_2(d_1 - 1) < 4d_1$ *we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \frac{2d_1d_2}{4d_1 - d_2(d_1-1)} \begin{pmatrix} 1 + \dfrac{1}{d_1} \\ 1 - \dfrac{2}{d_2} \\ \dfrac{4d_1 - d_2(d_1-1)}{-d_1d_2} \\ 1 + \dfrac{2}{d_1} + \dfrac{2}{d_2} \end{pmatrix}, \qquad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \alpha^{(4)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \dfrac{-d_1d_2}{4d_1 - d_2(d_1-1)} \\ \alpha^{(4)} \end{pmatrix},$$

$$\begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \frac{d_1d_2}{4d_1 - d_2(d_1-1)} \begin{pmatrix} \dfrac{2(1-n)}{d_2(d_1-1)} \\ \dfrac{d_1+1}{d_1-1} \end{pmatrix},$$

where

$$\kappa_1\alpha^{(1)}\alpha^{(3)} = (\alpha^{(2)})^2, \qquad \kappa_2\alpha^{(2)}\alpha^{(4)} = (\alpha^{(1)})^2.$$

(Ib) *If* $d_1(d_2-1) < 4d_2$ *and* $d_1 \neq 2$ *we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \frac{2d_1d_2}{4d_2 - d_1(d_2-1)} \begin{pmatrix} 1 - \dfrac{2}{d_1} \\ 1 + \dfrac{1}{d_2} \\ 1 + \dfrac{2}{d_1} + \dfrac{2}{d_2} \\ \dfrac{4d_2 - d_1(d_2-1)}{-d_1d_2} \end{pmatrix}, \qquad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \alpha^{(4)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \dfrac{-d_1d_2}{4d_2 - d_1(d_2-1)} \end{pmatrix},$$

$$\begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \frac{d_1d_2}{4d_2 - d_1(d_2-1)} \begin{pmatrix} \dfrac{n-1}{d_2(d_1-1)} \\ \dfrac{d_1-2}{d_1-1} \end{pmatrix}$$

with the same constraint relations for $\alpha^{(i)}$ *as in* (Ia).

(Ic) *If* $d_1 = 2$ *we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ -2 \end{pmatrix}, \qquad \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -1 \\ n_2 \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \alpha^{(4)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \dfrac{-d_2}{d_2+1} \end{pmatrix}, \qquad \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ \beta^{(2)} \end{pmatrix},$$

where $n_2 = 0$ *or* 1 *and the same constraint relations for* $\alpha^{(i)}$ *as in* (Ia) *hold. Also, we have* $\beta^{(2)} = \alpha^{(1)}$ *when* $n_2 = 1$ *and* $\beta^{(2)}$ *is arbitrary when* $n_2 = 0$.

(II) *In all cases we can have*

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \alpha^{(4)} \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \\ -\dfrac{1}{3}\left(\alpha^{(1)} + 2\alpha^{(2)} - \dfrac{n+d_2}{n-1}\right) \\ -\dfrac{1}{3}\left(2\alpha^{(1)} + \alpha^{(2)} - \dfrac{n+d_1}{n-1}\right) \end{pmatrix},$$

$$\begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{d_1-1} \\ \dfrac{-d_1}{d_1-1} \end{pmatrix},$$

where $\alpha^{(1)}$ *and* $\alpha^{(2)}$ *are uniquely determined by* $\tau = A_1\alpha^{(2)}/A_2\alpha^{(1)}$ *which must be a solution of the Einstein equation for G-invariant metrics on G/K.*

Outline of proof. Since the required analysis is very similar to that for Case (iii), we will only indicate the differences. First suppose that $m_1 = 0$. As before, $n_2 > -1$ and we may then conclude that $n_1 = -1$. However, this time we have $(m_2, m_3, m_4) = (2, 4, -2)\beta^{(1)}$ and we obtain a contradiction as before except when $\beta^{(1)} > 0$ and $d_1 = 2$. In this situation, the equations imply that $m_4 = -2$, $\beta^{(1)} = 1$ and $\alpha^{(4)} = -d_2/(d_2+1)$. Hence $m_2 = 2$, $m_3 = 4$ and we are in Case (Ic). The rest of the conclusions come from examining Eq. (3.14) further.

So if $d_1 \neq 2$, then $m_1 \neq 0$ and $n_1 = n_2 = -1$ as in Case (iii). We again conclude that m_1, m_2, m_3, m_4 are either all different or they are all equal. Case (Ia) corresponds to the situation when m_3 is the smallest among the m_i and Case (Ib) corresponds to the situation when m_4 is smallest. The Diophantine inequalities result from similar considerations as in Section 4. Notice that in Case (Ib) d_1 cannot be 2 because otherwise there would be an s^{-2} power on the left-hand side of (3.14) which is not balanced by any term on the right-hand side. Thus we see that Case (Ic) is really the complement of Case (Ib).

When all the m_i are equal, there are again 2 subcases, as in Case (iii). Either $m_i = -2$, which gives Case (II), or else $m_i < -2$.

In the former case, one easily checks that $\alpha^{(3)}$ and $\alpha^{(4)}$ can be expressed in terms of $\alpha^{(1)}$ and $\alpha^{(2)}$ as indicated above, while the latter constants must satisfy the system

$$(\alpha^{(1)})^2 + 2\alpha^{(1)}\alpha^{(2)} + \frac{3(\alpha^{(2)})^2}{\kappa_1} = \left(\frac{n+d_2}{n-1}\right)\alpha^{(1)},$$
$$(\alpha^{(2)})^2 + 2\alpha^{(1)}\alpha^{(2)} + \frac{3(\alpha^{(1)})^2}{\kappa_2} = \left(\frac{n+d_1}{n-1}\right)\alpha^{(2)}.$$

If we divide these equations by $(\alpha^{(1)})^2$, we see immediately that they say that $\alpha^{(1)}$ and hence $\alpha^{(2)}$ are determined by the ratio $\alpha^{(2)}/\alpha^{(1)}$, which by (9.1) and (9.2) is just the asymptotic value of $A_2 f_1^2/A_1 f_2^2$. As in Case (iii), the above system corresponds to a cubic equation in τ, which is precisely the Einstein condition for G-invariant metrics on the principal orbit G/K.

The latter case can occur only if

$$(9-\kappa_1\kappa_2)^2 = 4\kappa_1\kappa_2(\kappa_1-3)(\kappa_2-3)$$

holds. Then we must have $m_i = -2(d_1-1)\beta^{(1)} < -2$, $\beta^{(2)} = -d_1\beta^{(1)}$, $n_1 = n_2 = -1$ and

$$\alpha^{(1)} + 2\alpha^{(2)} + 3\alpha^{(3)} = 0, \qquad 2\alpha^{(1)} + \alpha^{(2)} + 3\alpha^{(4)} = 0,$$
$$(9-\kappa_1\kappa_2)\alpha^{(1)} - \kappa_2(2\kappa_1-6)\alpha^{(2)} = 0.$$

We will refer to this last case as Case (III). However, we have not listed this case in the statement of Theorem 8.1 because we shall show later that this set of leading terms does not give rise to convergent Painlevé expansions.

Remark 8.2. Note the appearance of the condition $d_2 < 4d_1/(d_1-1)$ (and the corresponding condition with d_1, d_2 interchanged) as in Case (iii). In particular, (d_1, d_2) satisfies both conditions iff $d_1 + d_2 \leq 8$, or $d_1 = 1$, or $d_2 = 1$, or $(d_1, d_2) = (7, 2)$ or $(2, 7)$.

Theorem 8.3. *The resonances in the Painlevé analysis of the system (3.9)–(3.14) subjected to the constraints (3.16) and (3.17) are as follows*:

Cases (Ia) *and* (Ib) $R = -1, 0, 1, 2$;

Case (Ic) $R = -1, 0, 2$, *where* -1 *has multiplicity* 2 *when* $n_2 = 1$ *and multiplicity* 1 *otherwise, and* 0 *has multiplicity* 1 *when* $n_2 = 1$ *and multiplicity* 2 *otherwise.*

Case (II) $R = -1, 2$ *and the roots of the equation*

$$R(R-1) = \frac{2}{1-n} - \frac{4n}{d_1 d_2}(\alpha^{(3)} + \alpha^{(4)}), \tag{8.1}$$

where $\alpha^{(3)}$ *and* $\alpha^{(4)}$ *are given in* Theorem 8.1(II).

The proof of this theorem proceeds in a similar fashion as in Section 5. We will summarise below some useful information from the proof and discuss the resonance analysis for the undisplayed set (III) of possible leading terms in Theorem 8.1.

For Case (Ia), $\ker(Z(R=1))$ is spanned by $(-4\alpha^{(1)}, -2\alpha^{(2)}, 0, -6\alpha^{(4)}, (d_1+1)/(d_1-1), -(2d_1/(d_1-1)))$, $\ker(Z(R=2))$ is spanned by

$$\left(\frac{d_2(d_1+1)(d_1-1)}{(n-1)d_1}\alpha^{(1)}, \left(\frac{d_2(d_1+1)}{n-1}-2\right)\alpha^{(2)}, \left(\frac{d_2(d_1+1)^2}{(n-1)d_1}-4\right)\alpha^{(3)},\right.$$
$$\left.\left(\frac{d_2(d_1+1)(d_1-2)}{(n-1)d_1}+2\right)\alpha^{(4)}, -2, \frac{d_2(d_1+1)}{n-1}\right),$$

$\ker(Z(R=2)^{\mathrm{T}})$ is spanned by $(0, 0, (4d_1-d_2(d_1-1))/2d_2, 0, -2d_1, -(d_1+1))$. As in Remark 6.2, we see that we can choose Q to be 1 except when $d_2=3, d_1\neq 3$, in which case we can take $Q=d_1+3$.

For Case (Ib), $\ker(Z(R=1))$ is spanned by $(2\alpha^{(1)}, 4\alpha^{(2)}, 6\alpha^{(3)}, 0, (d_1-2)/(d_1-1), d_1/(d_1-1))$, $\ker(Z(R=2))$ is spanned by

$$\left(\left(\frac{(d_1-2)d_2(d_1-1)}{(n-1)d_1}\right)\alpha^{(1)}, \left(\frac{(d_1-2)d_2}{n-1}+1\right)\alpha^{(2)},\right.$$
$$\left.\left(\frac{(d_1-2)d_2(d_1+1)}{(n-1)d_1}+2\right)\alpha^{(3)}, \left(\frac{(d_1-2)^2 d_2}{(n-1)d_1}-1\right)\alpha^{(4)}, 1, \frac{(d_1-2)d_2}{n-1}\right),$$

$\ker(Z(R=2)^{\mathrm{T}})$ is spanned by $(0, 0, 0, (4d_2-d_1(d_2-1))/2d_2, d_1, -(d_1-2))$. Again, we can take $Q=1$ except when $d_1=3, d_2\neq 3$ in which case we can take $Q=d_2+3$.

For Case (Ic), $\ker(Z(R=2))$ is spanned by $(0, \alpha^{(2)}, 2\alpha^{(3)}, -\alpha^{(4)}, 1, 0)$, which is simply the vector given in (Ib) with d_1 set to be 2. The $\ker(Z(R=2)^{\mathrm{T}})$ is spanned by $(0, 0, 0, d_2+1, 2d_2, 0)$. We can set Q to be 1 as well.

For Case (II) $\ker(Z(R=2))$ consists of vectors of the form

$$\left(\left(\frac{d_1-1}{d_1}\right)\zeta_2\alpha^{(1)}, (\zeta_1+\zeta_2)\alpha^{(2)}, \left(2\zeta_1+\left(\frac{d_1+1}{d_1}\right)\zeta_2\right)\alpha^{(3)},\right.$$
$$\left.\left(-\zeta_1+\left(\frac{d_1-2}{d_1}\right)\zeta_2\right)\alpha^{(4)}, \zeta_1, \zeta_2\right),$$

where $(\zeta_1, \zeta_2)^{\mathrm{T}}$ lies in the null space of the matrix

$$\frac{2}{d_1-1}\left(\frac{2n+d_1d_2}{n-1}-2\alpha^{(1)}-2\alpha^{(2)}\right)\begin{pmatrix} \dfrac{1-n}{d_2} & \dfrac{1-n}{d_1d_2} \\ 1 & \dfrac{1}{d_1} \end{pmatrix}.$$

So $\ker(Z(R=2))$ is two-dimensional if $2\alpha^{(1)}+2\alpha^{(2)}=(2n+d_1d_2)/(n-1)$, and otherwise is spanned by $(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, -1/(d_1-1), d_1/(d_1-1))$. Similarly, $\ker(Z(R=2)^{\mathrm{T}})$ consists of vectors of the form

$$\left((d_1-1)\zeta_2, \left(\frac{1-n}{d_2}\right)\zeta_1+d_1\zeta_2, \left(\frac{2(1-n)}{d_2}\right)\zeta_1+(d_1+1)\zeta_2,\right.$$
$$\left.\left(\frac{n-1}{d_2}\right)\zeta_1+(d_1-2)\zeta_2, \zeta_1, \zeta_2\right),$$

where $(\zeta_1, \zeta_2)^T$ lies in the null space of the transpose of the matrix above. So $\ker(Z(R = 2)^T)$ is two-dimensional if $2\alpha^{(1)} + 2\alpha^{(2)} = (2n + d_1 d_2)/(n - 1)$ and otherwise is spanned by $(1, 1, 1, 1, 2d_2/(n - 1), 2)$.

In Case (III), the recursion operator $Z(j/Q)$ is given by the matrix

$$\begin{pmatrix} \frac{j}{Q} & 0 & 0 & 0 & 0 & -2\left(\frac{d_1-1}{d_1}\right)\alpha^{(1)} \\ 0 & \frac{j}{Q} & 0 & 0 & -2\alpha^{(2)} & -2\alpha^{(2)} \\ 0 & 0 & \frac{j}{Q} & 0 & -4\alpha^{(3)} & -2\left(\frac{d_1+1}{d_1}\right)\alpha^{(3)} \\ 0 & 0 & 0 & \frac{j}{Q} & 2\alpha^{(4)} & -2\left(\frac{d_1-2}{d_1}\right)\alpha^{(4)} \\ 0 & \frac{n-1}{d_2(d_1-1)} & \frac{2(n-1)}{d_2(d_1-1)} & \frac{-(n-1)}{d_2(d_1-1)} & 0 & 0 \\ -1 & \frac{-d_1}{d_1-1} & -\left(\frac{d_1+1}{d_1-1}\right) & -\left(\frac{d_1-2}{d_1-1}\right) & 0 & 0 \end{pmatrix}.$$

The difference between this operator and that for Case (II) is that the zero 2×2 matrix in the lower right-hand corner is replaced by $(j/Q - 1)$ times the identity matrix. As a result of this difference, in the present case, for all $j > 0$, $Z(j/Q)$ has a one-dimensional kernel spanned by $(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \alpha^{(4)}, j/2Q(1-d_1), jd_1/2Q(d_1-1))$. Likewise, $\ker(Z(j/Q)^T)$ is spanned by $(1, 1, 1, 1, jd_2/Q(n - 1), j/Q)$.

Theorem 8.4. *Consider the system* (3.9)–(3.14) *with the constraints* (3.16) *and* (3.17):

1. *For leading terms of types* (Ia, b, c) *all compatibility conditions hold. Moreover, the Hamiltonian constraint* (3.15) *fixes the degree of freedom at the top resonance. Hence we have convergent Painlevé expansions depending on the full number of parameters* (3) *except in case* (Ic) *with* $n_2 = 1$, *when we have a 2-parameter family instead. Moreover, the expansions are meromorphic in s* (*rather than a fractional power of s*) *except in case* (Ia) *with* $d_2 = 3$, $d_1 \neq 3$ *and in case* (Ib) *with* $d_1 = 3$, $d_2 \neq 3$.
2. *For leading terms of type* (II), *if* $2\alpha^{(1)} + 2\alpha^{(2)} \neq (2n + d_1 d_2)/(n - 1)$ *and all earlier compatibility conditions hold, then the compatibility condition at the top resonance* $R = 2$ *holds as well, and the Hamiltonian constraint* (3.15) *fixes the degree of freedom at the top resonance*.
3. *For leading terms of type* (III), *the compatibility conditions fail to hold all the way*.

The proofs for parts 1 and 2 are similar to those for the corresponding leading terms in Case (iii). Below we will indicate the argument for part 3.

In writing out the recursion relations for leading terms of type (III), let $m = 2(1-d_1)\beta_0^{(1)}$ denote the common value of the exponents m_i. Then $-m-2$ equals j_0/Q for some positive integer j_0. We will show that the compatibility condition at $j = j_0$ cannot hold. The right-hand side of the recursion relation at step j is

$$\begin{pmatrix} \sum_1^{j-1} 2\left(\frac{d_1-1}{d_1}\right)\alpha_k^{(1)}\beta_{j-k}^{(2)} \\ \sum_1^{j-1} 2\alpha_k^{(2)}\left(\beta_{j-k}^{(1)}+\beta_{j-k}^{(2)}\right) \\ \sum_1^{j-1}\alpha_k^{(3)}\left(4\beta_{j-k}^{(1)}+2\left(\frac{d_1+1}{d_1}\right)\beta_{j-k}^{(2)}\right) \\ \sum_1^{j-1}\alpha_k^{(4)}\left(-2\beta_{j-k}^{(1)}+2\left(\frac{d_1-2}{d_1}\right)\beta_{j-k}^{(2)}\right) \\ \left(1-\left(\frac{j-j_0}{Q}\right)\right)\beta_{j-j_0}^{(1)} \\ \left(1-\left(\frac{j-j_0}{Q}\right)\right)\beta_{j-j_0}^{(2)} \end{pmatrix}.$$

In view of the expression of the generator of $\ker(Z(j/Q)^{\mathrm{T}})$, the compatibility condition at $j=j_0$ becomes

$$0=2\left(\frac{d_1-1}{d_1}\right)\sum_{k=1}^{j_0-1}\left(\alpha_k^{(1)}+\left(\frac{d_1}{d_1-1}\right)\alpha_k^{(2)}+\left(\frac{d_1+1}{d_1-1}\right)\alpha_k^{(3)}+\left(\frac{d_1-2}{d_1-1}\right)\alpha_k^{(4)}\right)$$
$$\times\beta_{j_0-k}^{(2)}+\sum_{k=1}^{j_0-1}2(\alpha_k^{(2)}+2\alpha_k^{(3)}-\alpha_k^{(4)})\beta_{j_0-k}^{(1)}+\left(\frac{j_0d_2}{Q(n-1)}\right)\beta_0^{(1)}+\left(\frac{j_0}{Q}\right)\beta_0^{(2)}.$$

If compatibility breaks down before $j=j_0$ then we are done. Otherwise, by the earlier recursion, the first two sums are zero and we are reduced to the condition

$$d_2\beta_0^{(1)}+(n-1)\beta_0^{(2)}=0,$$

which cannot hold since, from above, $\beta_0^{(2)}=-d_1\beta_0^{(1)}$, and $\beta_0^{(1)}\neq 0$ by assumption.

9. Asymptotics of the Painlevé solutions

Our Painlevé expansions give local solutions of the cohomogeneity one Ricci-flat equations, and the asymptotic behaviour near the movable singularity can be translated into corresponding behaviour of the Ricci-flat metrics. Recall that our metric $\bar{g}=\mathrm{d}t^2+f_1(t)^2B|_{\mathfrak{p}_1}+f_2(t)^2B|_{\mathfrak{p}_2}$, where $f_1^2=\mathrm{e}^{q_1}$ and $f_2^2=\mathrm{e}^{q_2}$.

It then follows that:

$$\mathrm{d}t=\left(\frac{z_1}{A_1}\right)^{d_1/2(n-1)}\left(\frac{z_2}{A_2}\right)^{d_2/2(n-1)}\mathrm{d}s,$$

and

$$f_1^2=\left(\frac{z_1}{A_1}\right)^{(1-d_2)/(n-1)}\left(\frac{z_2}{A_2}\right)^{d_2/(n-1)}, \tag{9.1}$$

$$f_2^2 = \left(\frac{z_1}{A_1}\right)^{d_1/(n-1)} \left(\frac{z_2}{A_2}\right)^{(1-d_1)/(n-1)}. \tag{9.2}$$

For leading term behaviour of type (Ia), we may assume that $s = 0$ corresponds to $t = 0$ and we have $t \sim s^{(4d_1+d_2)/(4d_1-d_2(d_1-1))}$. (We suppress multiplicative constants here and in what follows.) Furthermore, $f_1(t)^2 \sim t^{-(2d_2/(4d_1+d_2))}$ and $f_2(t)^2 \sim t^{4d_1/(4d_1+d_2)}$. Therefore, as t tends to 0, the volume of the principal orbits tends to 0 even as the $\mathfrak{p}_1$ directions blow up. Note that although the variables z_1 and z_2 do not blow up in this leading term behaviour, one of the geometric variables, f_1 in this case, does blow up.

For leading term behaviour of types (Ib) and (Ic), again we may assume that $s = 0$ corresponds to $t = 0$ and we have $t \sim s^{(d_1+4d_2)/(4d_2-d_1)(d_2-1)}$. Also, $f_1(t)^2 \sim t^{4d_2/(d_1+4d_2)}$ and $f_2(t)^2 \sim t^{-(2d_1/(d_1+4d_2))}$. Hence as t tends to 0, the volume of the principal orbits tends to 0, but this time the $\mathfrak{p}_2$ directions blow up.

Finally, for leading term behaviour of type (II), we may assume that $s = 0$ corresponds to $t = +\infty$ and we have $t \sim s^{-(1/(n-1))}$. Furthermore, $f_1(t)^2 \sim (n-1)^2(A_1/\alpha_0^{(1)})t^2$ and $f_2(t)^2 \sim (n-1)^2(A_2/\alpha_0^{(2)})t^2$. Hence $\bar{g}$ is asymptotic to the Ricci-flat metric cone on G/K with a G-invariant Einstein metric, and so has Euclidean volume growth.

The above analysis applies to both Cases (iii) and (iv), as well as to Case (i), provided that the particular leading term behaviour exists, because the transformation from the z_i to f_j depends only on the basis $\{w^{(1)}, w^{(2)}\}$ chosen and the leading term behaviour. Finally, we observe that as a result of the Painlevé analysis in this paper and in [6], it follows that the only Painlevé expansions among the two-summand case which exhibit a Taub-NUT-style end behaviour occur when one of the irreducible summands is Abelian (i.e. Case (ii) of Section 3).

Acknowledgements

The first author thanks John Wilson for valuable discussions on Diophantine equations. The second author is partially supported by NSERC grant No. OPG0009421.

References

[1] M. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type, I, J. Math. Phys. 21 (1980) 715–721;
M. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type, II, J. Math. Phys. 21 (1980) 1006–1015.

[2] M. Adler, P. van Moerbeke, Kowalewski's asymptotic method, Kac–Moody Lie algebras and regularization, Commun. Math. Phys. 83 (1982) 83–106.

[3] R. Bryant, S.M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989) 829–850.

[4] A. Dancer, M. Wang, Integrable cases of the Einstein equations, Commun. Math. Phys. 208 (1999) 225–244.

[5] A. Dancer, M. Wang, The cohomogeneity one Einstein equations from the Hamiltonian viewpoint, J. Reine Angew. Math. 524 (2000) 97–128.

[6] A. Dancer, M. Wang, The cohomogeneity one Einstein equations and Painlevé analysis, J. Geom. Phys. 38 (2001) 183–206.

[7] G.W. Gibbons, D. Page, C.N. Pope, Einstein metrics on $\mathbb{R}^3$, $\mathbb{R}^4$ and S^3 bundles, Commun. Math. Phys. 127 (1990) 529–553.
[8] G. Jensen, Einstein metrics on principal fibre bundles, J. Diff. Geom. 8 (1973) 599–614.
[9] S. Kowalewski, Sur le probleme de la rotation d'un corps solid autour d'un point fixe, Acta Math. 12 (1889) 177–232.
[10] M. Wang, W. Ziller, Existence and nonexistence of homogeneous Einstein metrics, Invent. Math. 84 (1986) 177–194.
[11] W. Ziller, Homogeneous Einstein metrics on spheres and projective spaces, Math. Ann. 259 (1982) 351–358.